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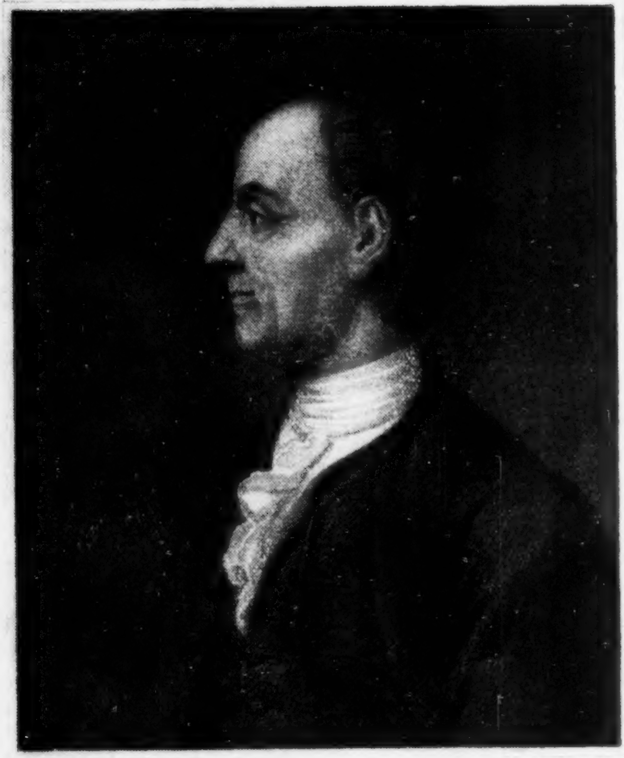

THE MATHEMATICS TEACHER

Volume XXVII — Number 4

Edited by William David Reeve

A Study of Prognosis of Probable Success in Algebra and in Geometry*

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INTRODUCTION

THE EVERPRESENCE of the problem of individual differences in school work and the importance of its solution in the training of the children entrusted to our care is again evidenced by statements that have appeared recently in the press. Dr. Eugene A. Colligan, Associate Superintendent of Schools in New York City, is quoted as saying before the 1933 Junior High School Conference in New York City:

> The educational doctrine of individual differences is an established truth. Not all pupils have similar capacities or needs. All pupils possessed of the same talent are not capable of the same degree of development. But the discovery of aptitudes and talents in each pupil is possible and is necessary.

An editorial in the New York Evening World-Telegram late in March 1933 contained the following statement:

* This is the first part of a discussion on "A Study of Probable Success in Algebra and Geometry." The second and last part will appear in the May Number of *The Mathematics Teacher*.

> The theory of education is that the public school should serve all the pupils, whether geniuses, mediocrities or morons. Under the present system only the majority is served. The schools do not properly provide for the especially apt or the slow ones. The result shows in the fact that more than a third of all the children entering high school are unable to do the work offered them. The new idea is to differentiate within the school system, to place pupils so that apt ones will not be held back by the slow ones nor the slow ones impressed with the consciousness of failure through seeing themselves outdistanced by children who move faster in the studies offered. . . . Teach children according to their needs and capacities. Reform the school system to suit that educational ideal.

Dr. William J. O'Shea, City Superintendent of Schools in New York City, in a recent report on curriculum for slow children says:

> The day is past when boys and girls are to be broken on the wheel of established school organizations, whether elementary or high, and the spirit of teachers crushed by condemning them to impossible tasks. Schools are justified in giving their diplomas to pupils who have reached, through honest effort, the limit of their development. . . . Boys and girls do not attend school to be fitted into rigid schemes of studies, but schemes and objectives must be fitted to the needs and abilities of children. . . . Unreasonable demands for achievement made upon teachers and pupils, which demands are preordained to failure, can beget only a feeling of helplessness and hopelessness in teachers and of opposition in pupils.

Dr. J. Cayce Morrison, Assitant State Commissioner of Education in New York State, said recently before the Academy of Public Education:

> The school's program must be adapted to the needs of individual children. . . . The theory of individual differences is the most potent idea in the remaking of the public school program. . . . The child must find expression for those talents within him. The habit of success must be an essential part of his training.

Previous Experiments and Studies

For the past fifteen years or more the classification of pupils has been considered one of the leading problems in high school administration. In the New York City school system, for example, the annual general circular from the office of the City Superintendent concerning graduation from the elementary schools and admission to high school has contained the following statement: "In conjunction with the teachers, principals are expected to make a careful study of each pupil's abilities and aptitudes, so that they may give intelligent advice to the parents as to the future education of their children."

A number of attempts has been made to base the classification

on the use of intelligence tests. Breed[1] reports that coefficients of correlation between intelligence tests and achievement scores range all the way from very low to very high and the more dependable ones seem to cluster somewhere between .40 and .60 with an error of classification of the pupils which runs as high as 50 per cent.

Fleming,[2] in her dissertation on a Detailed Analyses of Achievement in the high school, states as one of her conclusions:

> General intelligence, as measured by objective mental tests, showed a high correlation with achievement in the high school, even where the minimum level of ability is normal or average intelligence. . . . In comparison with character traits, intelligence appears the most significant factor for accomplishment in the senior high school, the correlation being $.6339 \pm .05$.

However, in her introduction she says,

> Investigators are perceiving more and more clearly that general ability, even though admittedly the most important single factor in school achievement, can be but one factor. The use of mental tests has shown objectively that perfect correlation cannot be expected. The habits and character of the pupil play a potent rôle in determining his achievement and in furnishing the teacher's estimate of his accomplishment.

John L. Stenquist reports a study with a group of eighty first term boys in the Commercial High School, Brooklyn, N. Y., to whom a series of intelligence tests were given. He states that:

> The boys as a group were exactly of average intelligence, that it was clearly shown that several of the boys had marked ability, not only in mechanical ability, but also in intelligence, that as compared with the men in the army who were clerks, this whole group ranked much higher and that other causes were apparently responsible for the failure of these boys in the first term of the high school.[3]

He concludes that the tests were inadequate and that the school was doing a serious injustice in forcing the pupils to attempt something in which they were predestined to fail.

In December 1919 the Otis Intelligence Test was given to 288 boys in the Boys High School, Brooklyn, N. Y.

> The correlation between the test scores and the midterm results at the end of the ten weeks was low. Of twenty-one boys with mental age less than thirteen, four failed no subject, six failed one subject, six failed two subjects, while of thirty-five boys with the best records in the test, seven were not fit for rapid advancement as

[1] Breed, F. S., *School and Society*, 1923.

[2] Fleming, Cecile W., *A Detailed Analysis of Achievement in the High School*. T. C. Contribution No. 196. 1926.

[3] *Bulletin of High Points*. Department of Education, New York City, April, 1922.

the work was done that term. Six of the seven failed in one subject and did mediocre work in the other two. Twenty boys in the original rapid advancement classes which had been organized the preceding September on the basis of elementary school records would not have been given the chance on their intelligence test records.[4]

In 1920 in the Manual Training High School, Brooklyn N. Y. 202 boys in the general course and 306 in the technical course were graded on the basis of the Otis Intelligence Test and were taught their four subjects by thirty-six different teachers. The latter, who did not know the position of the classes on the scale, were asked to rate each class they instructed as excellent, fair or poor. The report of the thirty-six teachers agreed with the position of the class on the Otis scale. The committee in charge concluded that the evidence submitted in the report tended to the conclusion that the Otis Group Intelligence Test afforded "a simple economical means for grading pupils entering high school in the terms of ability to do first grade high school work."[5]

At the end of the term in this experiment 46 per cent of the top Latin section failed in Latin and 20 per cent in algebra, while in the lowest Latin section 30 per cent passed in Latin and 35 per cent in algebra. The range of I.Q's in the former was from 131 upwards and in the latter from 90 to 95. These figures are typical of others that might be quoted from the French and Spanish groups. Of the thirteen classes in the technical course, the class that was second from the top on the Otis scale was ninth in the order of per cent passing all subjects, while the fourth section was first. The question may therefore, be raised as to what the grading on the basis of the intelligence test did for the 46 per cent that failed in Latin and the 20 per cent that failed in algebra in the highest group? Suppose that the school had decided not to permit group three in the general course to study a foreign language. Would not an injustice have been done to the 30 and 41 per cent that received credit for Latin and for French? In the technical course, why should group two have been ninth in order of passing all subjects, when group nine was the tenth? It is difficult to see how these figures justify the conclusion reached by the committee. These questions and others like them can be answered best by concluding from the table the existence of a condition with which every teach-

[4] *Op. cit.*, February, 1920.
[5] *Op. cit.*, April, 1920.

er is acquainted namely, that in every carefully selected bright class there are, for a given subject, pupils who seem to be out of place. In the slow classes also one finds a few who are above the low level of their classmates. We are interested here not so much in the pupil's general ability as in his ability to profit by instruction in the particular subject he is studying. Intelligence tests were not meant to be employed as the sole basis of classifying pupils. Success in school work is not quantitatively coextensive with intelligence alone, but with intelligence in conjunction with traits that are not evaluated by the intelligence tests.

In one of the most recent studies of the problem of grouping, Keliher writes:[6]

> In the actual use of intelligence tests the purpose of providing for individual differences is often defeated, for the results of the tests are used in terms of Mental Age and the I.Q. These are usually averages of separate categories within the test and, as averages, tend to obscure differences in mental functions. This is illustrated by the fact that children may achieve the same mental age by any of the many combinations. . . . This makes for a relatively high degree of specificity of mental abilities. Certainly it seems true that one cannot in any general sense prescribe a specific educational program for an individual on the basis of the I.Q. or M.A., when these indices themselves are averages of specific differences which may be the very matters that should be brought into consideration. . . . A general "homogeneity" does not exist naturally. Homogeneities are specific. The specific nature of abilities and natural homogeneities makes such a general homogeneity impossible of natural achievement.

Another recent study by Burr reports that "in general, after grouping has been carried on, four-fifths of the total ranges of ability in the original undivided group remains in each of the subdivided homogeneous groups."[7] Hollingshead in 1928 had likewise reported great overlapping of abilities in his attempt to find adequate bases for homogeneous grouping.[8]

The whole matter of the use of the intelligence tests as a basis for classification has become a highly controversial issue, in which leading educators have taken sides. It seems to have resolved itself

[6] Keliher, Alice V., "A Critical Study of Homogeneous Grouping in the Elementary Schools." Teachers College, 1930. Pp. 16 and 22.

[7] Burr, Marvin. "A Study of Homogeneous Grouping in Terms of Individual Variations and the Teaching Problem. A Dissertation in Manuscript." Teachers College, Columbia University. 1930.

[8] Hollingshead, A. D., "An Evaluation of the Use of Certain Educational and Mental Measurements For Purposes of Classification." T. C. Contribution No. 302. 1928.

into a question of general ability versus specific ability in specific subject matter.[9]

Several studies have also been made of the classification of pupils on the basis of grade school records, in order to determine to what extent previous scholastic success can be used for predicting academic success. Ross reports[10] correlations ranging from .36 to .56 between grade school record and average for the first year in high school, from .38 to .54 between grade school record and marks in English, from .25 to .54 between grade school record and marks in Latin and from .26 to .42 between grade school record and marks in algebra. He concludes that "correlations between grade school records and high school achievement are sufficiently high to be significant, being in most cases higher than the corresponding correlations between standard test scores and high school achievement."

Knowledge of the correlation between two traits enables one to judge the amount of one from an amount of the other. The higher the correlation the greater the accuracy of judgement. The most practicable way to consider the predictive value of a coefficient of correlation is probably the standard error of estimate, $\sigma\sqrt{1-r^2}$. According to Garrett,[11]

> This follows from the fact that $\sigma_{est.}$, which enables us to tell how accurately we can estimate an individual's score on test X_1 knowing his score on test X_2, depends on the r between the two tests. When r equals 1.00, $\sigma_{est.}$ is zero, which means that we can predict a score in X_1 from a knowledge of X_2 with perfect accuracy. When r equals .00, $\sigma_{est.}$ equals σ_1 directly, which means that we can be certain only that the predicted score lies somewhere within the limits of the X_1 distribution. As r decreases

[9] There are other studies also of homogeneous grouping, to which the reader may refer, such as,

Purdom, T. L., "The Value of Homogeneous Grouping." University of Michigan Research Monograph No. 1. Warwick and Yorke, Baltimore, Md. 1929.

Cook, R. R., "Results of Homogeneous Grouping in High School." Twenty-third Year Book of the National Society For the Study of Education. Part I, 302–312.

Billet, R. O., "A Controlled Experiment to Determine the Advantages of Homogeneous Grouping." *Educational Research Bulletin*, Ohio State University. April, 1928.

Schinnerer, N. C., "Status of Classification in the Cleveland Junior and Senior High Schools." Bulletin No. 59. Cleveland Bureau of Educational Research, 1929.

[10] Ross, C. C., "The Relation Between Grade School Records and High School Achievement." T. C. Contribution No. 156. 1925.

[11] Garrett, Henry E., *Statistics in Psychology and Education*. Longmans Green and Co., New York, 1926. Page 288.

from 1.00 to zero, the standard error of estimate rapidly increases, so that predictions from the regression equation range all the way from certainty to practically guesswork. The closeness of the correspondence denoted by r, therefore, may be guaged by the size of $\sigma_{est.}$

In fact the predictive value or r may be estimated from $\sqrt{1-r^2}$. alone, since this expression gives the proportionate amount of reduction in the standard error of estimate as r varies from .00 to 1.00. Since this is so, one wonders how significant are correlations that vary from .26 to .42 in the case of algebra, especially since even a value of .50 for r makes the prediction only sixteen per cent better than a guess.

Fretwell[12] experimented with two hundred boys who had come to the Speyer Junior High School, New York City, from five elementary schools. He used eleven standardized educational and psychological tests and took into consideration the judgment of four teachers after teaching the pupils for one year. He says nothing however, about success in individual subjects. This, after all, is the important thing. He furnishes no information about the nature of the work done by each pupil in Latin, Algebra, etc. He found the correlation of marks in the first year at Speyer with marks the year previous to coming to Speyer to be .42, with all marks before coming to Speyer .49, with a composite of eleven tests .56, and the correlation between marks for six years in the elementary school and the ranking of four teachers at the end of one year in the high school .50. He concludes from these that the tests were a better means of prognosis for these pupils than all their previous school marks.

In September 1925 Jacob S. Orleans and Michael Solomon[13] gave a Latin prognosis test to over 350 pupils in eight schools in the state of New York, in order to determine to what extent a test, composed of simple Latin lessons stressing the simpler Latin processes and using Latin content, would foretell the achievement of the pupils in Latin during the first half of the year. The test was given before the pupils had received any instruction in the work of the term. The criterion employed at the end of the term was the average between the teachers' ratings and the scores on an objective test in Latin, later used in a survey of achievement in

[12] Fretwell, E. K., "A Study in Educational Prognosis." T. C. Contribution No. 99. 1919.

[13] From an unpublished report by the authors.

Latin in New York State. The correlation between the prognosis test scores and the criterion varies from .74 to .85. The authors concluded that "this was fairly high especially in view of the fact that only one factor, the most potent one, is measured here, namely, the ability to handle Latin situations similar to those involved in learning Latin in the class room."

This study also offers some data on the prognostic value of intelligence tests. In one school forty pupils who took the prognosis test also took two intelligence tests, the Terman Group Test of Mental Ability and the Otis Self-Administering Test. For the former the correlation was .45 between the intelligence test scores and the measures of achievement in Latin. For the latter the corresponding correlation was .27. In another school similar data for 103 pupils gave a correlation of .41. The authors conclude that, in general, correlations between intelligence test scores and marks in high school subjects are between .30 and .60 and that is too low for purposes of estimating success in a school subject. An intelligence test used alone is certainly unsatisfactory.

The first attempt to measure mathematical ability was made by Rogers in 1916,[14] when she experimented with two groups of pupils in New York City, one of 53 girls in Wadleigh High School and the other of 61 in the Horace Mann School for Girls. The children were given a battery of tests that involved arithmetic, algebra, geometry, and four language tests, mixed relations, logical opposites, Trabue language scales and the Thorndike Reading Tests. These were given to pupils who had studied a certain amount of high school mathematics. From these Rogers finally chose a sextet that composes what is known now as the Rogers Test of Mathematical Ability. Rogers expressed her purpose as being "to make an analysis of the abilities involved in high school mathematics, to determine their efficiency and status, their interrelations and their connection with certain forms of mental capacity." She found the correlation between the school marks and the test scores to be .70 for the Wadleigh group and .88 for the Horace Mann group. Combining the two and giving the Horace Mann results double weight because of their greater reliability, she found the correlation between school marks and the composite for mathe-

[14] Rogers, Agnes L., "Experimental Tests of Mathematical Ability and Their Prognostic Value." T. C. Contribution, No. 189. 1918.

matics ability to be .82. In April 1923 Rogers listed in *The Mathematics Teacher* the results of the use of her test in ten schools in various parts of the country. The size of the groups varied from 26 to 115 and the correlations ranged from .34 to .76, six of them being below .60. The following statement from her article should be noted:[15]

> A satisfactory coefficient of correlation between a reliable criterion of success and a prognostic test, where the purpose in view is the placement of pupils in groups of equal ability, should be greater than .7. The low correlations in the table merely warn us against relying upon the test alone or on school marks alone. . . . It is very significant that there is no correspondence between intelligence test results and the results of the tests of geometric ability. It is apparent that geometric ability fails to be measured by tests of general ability. We need a new series of mathematical tests requiring innate talent. The abilities involved in these are intuitive and, therefore, likely to prove prognostic to a higher degree than the tests in the present sextet.

From a survey of the literature that is concerned with the problem of prognosis and classification, it seems that prognosis studies have been of three types, namely, (1) those based on the I.Q. or the elementary school marks, (2) those that are analytic of the skills involved and (3) those that give preliminary learning tests. A number of examples of type (1) were cited above. Type (2) is illustrated in the study of mathematical ability by Rogers and in the one by Kelly, described in his dissertation on Educational Guidance,[16] one of the objects of which he stated as "an analysis of the demands of certain high school courses." The third type is illustrated by the Orleans-Solomon study referred to above and that of prognosis in foreign languages by Symonds, in which he used tests of ability with the forms and grammar of the English language and quick-learning tests in a new language.[17]

The study conducted by the writer is the first to deal with prognosis in high school mathematics, other than on the basis of the I.Q., since 1916 when Rogers developed her tests of mathematical ability. While the latter were mainly concerned with an analysis of certain skills involved in high school mathematics, the present study is a combination of types (2) and (3), since the pupils are

[15] Rogers, Agnes L., "Tests of Mathematical Ability." *Mathematics Teacher*. June, 1923.

[16] Kelly, T. L., "Educational Guidance." T. C. Contribution No. 71. 1914.

[17] *Prognosis Tests in the Modern Foreign Languages*. Publication of the American and Canadian Committees on Foreign Languages. Vol. XIV. Macmillan, 1929.

given preliminary learning tests which are analytic of the skills involved.

Factors Involved in Prognosis in Algebra and in Geometry

Hull[18] distinguishes between tests that are designed to detect specific or particular aptitudes and those designed to detect general or average aptitudes. The former are exemplified by Patten's test for capacity to learn to operate the engine lathe, or the Orleans Solomon Latin Prognosis Test; the latter by the Stenquist Mechanical Aptitude Tests and also the numerous intelligence tests. In reality these are essentially tests of general or average scholastic aptitude as contrasted with tests for specific scholastic aptitude, which are necessary in our present elective system. The use of a specific prognosis test in a particular subject is based upon the belief that we have a large number of more or less specialized potential aptitudes or intelligences. A general intelligence test could be nothing but a kind of average of all potential aptitudes for any given person. On the other hand, the work in a school subject is very specific. In Latin, for example, it is entirely verbal and involves no numerical or spatial elements, and the verbalness is of a particular kind with its own organization. Algebra, however, is chiefly numerical, but of a symbolic type which makes it different from the languages and even from geometry, which is spatial rather than numerical and which involves a different type of symbolism.

The prognosis tests used in this study[19] are, therefore, ability tests in the sense that they measure the pupils' ability to do the type of work to be learned in the way in which it is to be learned. The pupil is given a little algebra and a little geometry in miniature, before he has received any instruction at all in the subject, and the purpose of the investigation is to see if the pupil's success in a period of eighty or ninety minutes will prognosticate success over a period of five months. An algebra test of this sort should take into account those factors that influence success in algebra achievement, namely, ability to handle algebra situations such as the pupil meets in the algebra classes and mastery of such pre-

[18] Hull, Clark L., *Aptitude Testing.* World Book Co. 1928.

[19] *The Orleans Algebra Prognosis Test,* World Book Co. 1928. *The Orleans Geometry Prognosis Test,* World Book Co. 1929.

requisites as arithmetic and reading with comprehension; and a geometry test of this sort should take into account ability to handle situations such as the pupil must meet in the study of the subject, a knowledge of algebra and such other information as is necessary for learning geometry by the methods now in common use and also skill in applying such information.

What factors are involved in a pupil's ability to cope with algebraic situations? Nunn[20] lists four important elements in the study of algebra as "analysis, the direct use of symbolism, the extended use of symbolism, and the manipulation of symbolism." Concerning analysis he says,

> A small boy has learnt that the area of a figure is the number of unit squares (say, square inches) which would entirely cover it. With this definition before him, he is asked to determine the area of a rectangle measuring seven inches by five. He soon observes that the unit squares into which the figure is mapped out can be regarded as forming five rows, each containing seven squares. This observation enables him to shorten the process of finding the area; for it is obvious that the rectangle must contain 7×5 or 35 square inches. So far arithmetic. But let the boy's attention shift from the actual manipulation of the figures to the process which the manipulation follows, and let him observe that the essence of that process is the multiplication of the length of the rectangle by the breadth. At this moment he has crossed the frontier which separates arithmetic from algebra; for it is an important part of the business of algebra to disengage the essential feature of an arithmetic process of a given type from the numerical setting which a particular case presents.

A great deal of the work done in the algebra class is of the sort described here, whether it be the translation of phrases into algebraic shorthand or of word problems into algebraic equations. In order to do this, one must have a sufficient command of the necessary symbolism; and in order to grasp the various topics that make up the content of the algebra course, one must have a facility in manipulating the symbols.

Thorndike also lists some of the abilities necessary in the study of algebra:[21]

> To understand formulas, to evaluate a formula by substituting numbers and quantities for some of its symbols, to rearrange a formula to express a different relation, to compute with line segments, angles, important ratios and decimal coefficients, to understand simple ratios, to construct graphs from tables of related

[20] T. Percy Nunn, *The Teaching of Algebra Including Trigonometry*. Longmans Green and Co. 1919.

[21] Thorndike, Edward L., *The Psychology of Algebra*. Macmillan Co., New York. 1928.

values and to understand the Cartesian coordinates, so as to use them in showing simple relations of y and x graphically.

Fundamentally Nunn and Thorndike agree about the abilities that a study of algebra seeks to establish and to improve. In order to acquire these broader abilities, the pupil must first master a number of details, some of which are (1) the notion that letters are used to represent numbers, (2) the substitution of numbers for letters, (3) the meaning and use of the exponent, (4) the understanding and the use of the directed numbers and (5) the translation into algebraic symbols of simple English phrases and statements. The writer's experience of twenty years in the teaching of algebra leads him to feel that an aptitude test of the sort used in this study should present the pupil with a larger number of simple abilities rather than a few broader ones to which he may not do justice in the few minutes allowed for each part of the test. If the pupil shows that he is able to master the tasks set because he has the abilities, the rest depends upon the learning process which, in turn, is closely linked with the method of presentation by the teacher. In order that these abilities may be discovered, the simplest thing seems to be to submit to the pupil some material in very elementary form which utilizes the learning process. For example, early in the year the algebra teacher finds it necessary to teach his pupils the meaning of various English expressions which they will need in the solution of problems, such as, 'one number exceeds another,' 'the excess of one number over another,' the difference between two numbers,' 'one number is greater than another by a certain amount.' He explains and illustrates the meanings of these terms and continues with a short effective drill. He also refers the class to the textbook and assigns the lesson to be done during the study period in school or at home. In doing the assignment the pupil goes through the same process again of studying an explanation and illustrations thereof and then working out some exercises based on them. A test composed of brief lessons and exercises following the same plan therefore is the instrument that calls for work similar to that which the pupil will be expected to do throughout the year.

Concerning the processes involved in the teaching of geometry, Judd says,[22]

[22] Judd, Charles H., *The Psychology of High School Subjects*. Ginn and Co., New York. 1915.

There are two problems involved in a geometrical demonstration. First, there is the problem of making sure that the student has clearly in mind the figure which is being discussed; second, there is the problem of studying the figure. If the student does not know the figure or if he cannot hold it in mind, he cannot perform later the more complex mental operations of dealing with the figure. If one has the figure in mind, then the real business of the science of geometry can begin. . . . Words help the student to carry his analysis beyond the figure as it is presented to his senses. The use of words in geometry is an example of one of these higher forms of thought. . . . The scientific investigation of the characteristics of figures leads us into the most elaborate logical processes of comparison and inference.

Here, just as in the case of algebra, in order to attain these broader abilities, the pupil must first master a number of smaller details, many of them mechanical. And so, again as a result of long experience in the classroom and of a great deal of thought, the writer feels that the study of geometry involves as basic elements the ability (1) to understand the axioms and to apply them to geometrical relationships, (2) to master the vocabulary of geometry, (3) to read geometric figures and references to them, (4) to represent statements geometrically, (5) to interpret geometric relations and (6) to solve geometric problems. A test based on these fundamental abilities in geometry, constructed like the algebra test, presents the pupil with the situation like the one he will meet in the classroom.

The tests used in this study, aiming at specific abilities, are, therefore, composed of a number of parts covering definite topics in the algebra and in the geometry. Each part consists of a brief lesson explaining and illustrating one small unit of work followed by a test based on the lesson, both lesson and test being carefully timed.

The writer checked his own judgment about the basic elements in the study of algebra and of geometry after the test had been administered by computing the correlations between each part of the test and marks representing achievement at the end of the term. The coefficients of correlation were such as to make the writer feel that he had made a satisfactory choice of topics for the various parts of the tests.

In order to be able to understand such printed material or the wording of problems that form part of the work, the pupil must be able to read with comprehension. In addition to the purely algebraic or geometric abilities, one must also consider whether or not the pupil is able to use the necessary arithmetic which is pre-

supposed. There is the ever present cry among the teachers of algebra that their pupils are not well prepared in arithmetic. They cannot be expected to continue the arithmetical operations in generalized form, when they do not know the operations themselves. If a deficiency in arithmetic or in reading is to be the cause of failure in algebra, a low score on a test of the sort used ought to enable one to predict such a failure.

In the study of both algebra and geometry, achievement is influences also mainly by three other factors, namely, intelligence, school habits and environmental conditions. There is no doubt in the minds of people who are acquainted with the practical situation in the classroom, that, in general, intelligence and ability in mathematics go together, although one meets very frequently a pupil of fine intelligence who finds the work beyond him. Concerning this Thorndike says,[23]

> Let not this emphasis on the extreme cases blind one to the main fact that, by and large, high intelligence means fine ability in algebra and low intelligence means poor ability in algebra. Of the twenty-two brightest pupils out of one hundred, seventeen will be above the average in algebra and of the twenty-two stupidest pupils, seventeen will be below the average in algebra.

The same can be said of geometry. The writer is interested particularly in the other five in each of these two groups. If the purpose of prognosis in algebra and in geometry is to predict the probable success or failure so that the pupils may be advised as to the selection of studies, emphasis must be placed on the individual case and not on the condition in general as portrayed in a coefficient of correlation.

A number of studies have been made concerning the relationship between intelligence and achievement in mathematics. Crathorne[24] reports a correlation of .50 between algebra marks and an average of two intelligence tests. Buckingham[25] tells of a correlation of .38 between algebra ratings and the score on the Army Alpha Test. Proctor[26] reports a correlation of .46 between algebra

[23] Thorndike, Edward L., *The Psychology of Algebra*. Macmillan and Co., New York. 1928. P. 426.

[24] Crathorne, A. R., "The Theory of Correlation Applied to School Grades." National Committee Report. 1923.

[25] Buckingham, B. R., "Mathematical Ability As Related to Intelligence." *School Science and Mathematics*. xxi, 205–215.

[26] Proctor, W. A., "Psychological Tests and Guidance of High School Pupils." *Journal of Educational Research*. Monograph 1. 1921.

marks and the Stanford Binet I.Q. Thorndike[27] reports correlations ranging from .47 to .53 between the algebra part in his intelligence test for high school graduates and the whole examination. Brooks[28] reports eight correlations ranging from .34 to .54 between algebra marks and an intelligence test and one of .51 between geometry marks and an intelligence test. Todd[29] reports a correlation of .39 between marks in algebra and scores on the Terman Group Test of Mental Ability and .33 between marks in geometry and the intelligence test.

In this study not much emphasis is placed on the correlation between marks in achievement and scores on the intelligence test. While achievement in the subject matter must be the criterion for the evaluation of any prognosis test, the writer feels that his purpose is to consider the pupil's ability as such and the relationship between achievement and specific ability rather than between achievement and general intelligence.

While a pupil's habits in regard to school work, such as attentiveness in the classroom, regularity of homework, perseverance, industry and the like, to a certain extent will determine the nature of his achievement in algebra and in geometry, they do not influence his inherent ability to master the subject. What the child is able to do, as contrasted with what he actually does, is not affected by these school habits. Nevertheless, in order to have some information about this factor in the situation, the New York Rating Scale for School Habits [30] was used with some groups. The scale includes the following traits: attention, neatness, interest, ambition, persistence, honesty, initiative, reliability, stability. The name of each trait was printed on one side of the page and after each trait a line on which were marked points representing a scale from zero to ten. In rating a pupil, the teacher indicated on the line after each trait the extent to which she judged the pupil as possessing the trait. In the group referred to, the pupils were rated by three teachers of the subjects they were studying that term other than

[27] Thorndike, E. L., "Abilities Involved in Algebraic Computation and in Problem Solving." *School and Society*, xv, 191–193.

[28] Brooks, Fowler D., *The Psychology of Adolescence*. Houghton Mifflin Co. 1929.

[29] Todd, Gertrude G., *Bulletin Department of Secondary School Principals*. 24: 74–76, January, 1929.

[30] Published by the World Book Co., Yonkers, N. Y.

algebra. The sum of the scores was considered a measure of each pupil's habits in school work.

The factor of environmental conditions is not treated as part of this study, because the author feels that the number of hours a pupil works after school or the occupation of his parents or the facilities for study at home etc. do not affect his ability to do algebra or geometry. A study by Clem[31] has shown that such factors are not significant in the case of Latin.

In evaluating a prognosis test one must keep in mind also the influence of the teacher upon the group. Teachers and supervisors of experience know that the same group of pupils will show different achievement with teachers of varying ability. The better teacher, through his skill, will probably get from the pupils work that approximates more closely their actual ability than will the relative achievement of pupils taught by a poor teacher.

[31] Clem, O. M., "Detailed Factors in Latin Prognosis. T. C. Contribution No. 144. 1924.

The Incommensurables of Geometry*

By E. T. Browne

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It is not known when man first began to count. Even the lower animals have a so-called ***number sense*** which enables them to distinguish certain numbers from others. For example, it is said that a crow can distinguish three from four; and as the German mathematician Hanckel puts it "Even a duck can count her young." Certain it is that as far back as we have any records, man has known how to count. From this process of counting there arose the positive integers, or natural numbers.

The introduction of fractions obviously came at a later date, although no one knows when that was. One of the oldest mathematical documents of which we have any record contains mention of them. This document, known as the Rhind Papyrus, has been dated by the archaeologists at about 1700 B.C. and is said to be founded upon a much older work. Its title is "Directions for obtaining a knowledge of all dark things." In it we find the problem:—Heap, its seventh plus its whole, it makes 19.

For centuries, in fact until about 500 B.C., these numbers, i.e., the positive whole numbers and fractions, sufficed for the needs of mankind. Man knew of no other sorts of numbers, nor did he feel the need of them. With these numbers he developed a remarkable system of numeration. And then came Pythagoras and the Pythagoreans. At the advice of Thales Pythagoras sojourned for many years in Egypt and from the Egyptians he may have learned of the "golden triangle," i.e., the right triangle whose sides are in the ratio 3:4:5. That such a triangle is a right triangle may possibly been known to the Egyptians. Perhaps their harpedonaptae or *rope stretchers* had made use of it in orienting their temples. Soon other Pythagorean triangles were discovered such as the 5:12:13 triangle and the 8:15:17 triangle. Tacitly they assumed that all triangles are rational, i.e., that whatever be the lengths of the sides

* An address delivered before the Mathematics Section of the South Piedmont District Teachers, The North Carolina Education Association, at Charlotte, N. C., Friday, Oct. 28, 1932.

of the triangle, there exists a unit of measure which may be applied exactly an integral number of times to each side. The fact that some of their triangles did not yield such perfect ratios did not surprise them. They felt that the fault was in not having chosen a sufficiently small unit of measure.

So matters stood for a while. Filled with the speculative spirit that then pervaded the Greek mind, Pythagoras attempted to discover some unifying principle in the universe. He became convinced that it was in numbers and their relations that was to be found the basis of true philosophy, and he proceeded to trace the origin of all things to number. The Pythagoreans attributed extraordinary properties to certain particular numbers. Thus, one is the essence of all things; four is the most perfect number and was in some mystic way thought of as corresponding to the human soul. Five is the cause of color, six of cold, seven of health and light, eight of love and friendship, etc.

The Pythagoreans paid much attention to the theory of proportion. By the aid of geometry, as was done in the fifth book of Euclid, the Pythagoreans developed a remarkable theory of proportion similar to that now given in texts on plane geometry.

The contemplation of right triangles soon led to a profound discovery, viz., the so-called Pythagorean Theorem. According to legend, this theorem was discovered by Pythagoras himself, who was so elated that he sacrificed an ox to the immortal gods. Some historians go so far as to say that in his enthusiasm he sacrificed a hecatomb or 100 oxen. It is doubtful that either of these stories is true, but at least, they serve to impress us with the fact that Pythagoras considered the theorem to be one of far-reaching importance, as indeed it is.

But the triumph was short lived, for one of its immediate consequences was another fundamental discovery, "The diagonal of a square is incommensurable with its side." This discovery came as a bomb-shell in the camp of the Pythagoreans. They who had boasted of the perfection and power of numbers were now confronted with lengths which could not be measured in terms of the numbers that they knew. The theory of proportion which had been worked out in such detail and with such perfection fell to the ground, since it was based on the false assumption that any two lengths are commensurable. The very name given to these incommensurables bears witness of the consternation that they

brought. Alogon, the unutterable, is what they were called and it is not strange that Pythagoras should have pledged his followers to secrecy until he could investigate more carefully this new phenomenon which had presented itself to him. Legend has it that the Pythagorean who divulged the secret of the incommensurable was drowned at sea.

What is this incommensurable that caused so much worry? And what do we mean when we say that the diagonal of a square is incommensurable with its side? Most of the modern texts on plane geometry merely make the statement that two magnitudes are incommensurable when there is no third magnitude of which both of them are multiples. Some books add that for example $2/7$ and $13/5$ are commensurable while $\sqrt{2}$ and 3 are incommensurable. In the former case it is easy to see that the two magnitudes are commensurable, for if we write them in the form $10/35$ and $91/35$ it is obvious that $1/35$ may be taken as the unit of measure and that this unit of measure is contained 10 times in the first quantity and 91 times in the second. In fact, it is easy to see that any two fractions are commensurable. Thus, if a/b and c/d are two fractions, i.e., a, b, c, and d are integers, we may write

$$a/b = ad/bd, \qquad c/d = bc/bd,$$

so that the magnitude $1/bd$ is contained an integral number of times in each.

Conversely, if two magnitudes a and b are commensurable, their ratio must be a fraction. For if e is the common unit of measure, there exist integers c and d such that

$$a = ce, \qquad b = de$$

whence

$$a/b = c/d.$$

The proof that the diagonal of a square is incommensurable with its side is so simple that most any schoolboy can understand it. Strangely enough, however, most textbooks on plane geometry do not attempt to give any proof but are content merely to make the statement "Thus $\sqrt{2}$ and 1 are incommensurable." To the student this statement sounds reasonable enough, but it is by no means obvious. When we reflect on it, is it self evident that $\sqrt{2}$ cannot equal to some fraction? The square of $7/5$ is very little less than 2,

while the square of 17/12, i.e., 289/144 exceeds 2 by only 1/144. Of all the infinity of fractions is it not possible that there is some one whose square is exactly equal to 2? No, there is no fraction whose square is exactly equal to 2. The proof that I shall give is found in the tenth book of Euclid, "which," Bertrand Russell facetiously remarks, "is one of those books that schoolboys supposed to be fortunately lost in the days when Euclid was still used as a text-book." The proof is based on the reductio ad absurdum and is thought by some to be Pythagoras' own proof. Simple as it is, I must confess that I had taken my Bachelor's degree and had entered a graduate school before I ever saw a proof.

Consider a square whose side is 1. If then x represents the length of the diagonal, we have by the Theorem of Pythagoras

$$x^2 = 2.$$

Now if x is commensurable with 1, we have just shown that the ratio of x to 1, i.e., x itself, must be a fraction. Suppose then that $x=p/q$ where p and q are integers such that the fraction is in its lowest terms, i.e., p and q have no common factor other than 1. We then have

$$\frac{p^2}{q^2} = 2, \qquad \text{or } p^2 = 2q^2.$$

Since p^2 is equal to an even integer, p itself must be even for the square of an odd integer is odd. If we write

$$p = 2p',$$

square and divide by 2, we obtain

$$2p'^2 = q^2.$$

Hence, by the argument just advanced, q must be even. But we have just shown that p is even, and this contradicts our assumption that p/q is a fraction in its lowest terms. That is, the assumption that $\sqrt{2}$ is equal to a fraction leads to a contradiction.

In a similar manner another Pythagorean, Theodorus of Cyrene, showed that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\cdots$, $\sqrt{17}$ are incommensurable with the unit. Such numbers are now called *irrationals*, in contradistinction to the fractions or *rational* numbers. When we attempt to express a fraction as a decimal, the process either terminates such

as $3/8 = .375000\ldots$ or else the decimal begins to repeat in blocks such as

$$2/11 = .181818 \cdots$$
$$2/7 = .285714 \,|\, 285714 \,|\, \cdots .$$

On the other hand, when we attempt to express an irrational as a decimal the result is never ending and non-repeating. Thus, $\sqrt{2} = 1.414213\ldots.$

Perhaps the most famous of all the irrationals is the number π which expresses the ratio of the circumference of a circle to its diameter. From time immemorial men have labored in vain to find an exact value for this ratio, and it is only with comparatively recent times that it has been shown that an exact value cannot be found.

In the Rhind Papyrus (1700 B.C.) π was taken to be $(16/9)^2 = 3.1604\ldots$, a very fair approximation. About 200 B.C. Archimedes, by considering regular inscribed and circumscribed polygons of 96 sides, showed that π lies between 3 1/7 and 3 10/71. We use the former number as an approximation even to this day. Quite a remarkable approximation is $355/113 = 3.14159203\ldots$ given by the Chinese Tsu-Ch'ung-chih (470 A.D.).

But none of these values is correct. For it was proved conclusively by the German mathematician Lambert in 1761 that π is irrational, i.e., cannot be expressed exactly as a fraction or as a decimal. In 1874 William Shanks computed π to 707 decimal places. Eight years later the German Lindemann proved that π is not only irrational but is also transcendental, i.e., cannot be a root of an algebraic equation with whole numbers as coefficients. It was this that stamped forever as vain the efforts of the circle-squarers.

Another famous irrational is the so-called e, approximately $2.71828\ldots$, the base of natural or Napierian logarithms. Its value has been computed to 225 places of decimals, but like π, e also is transcendental. It is now known that nearly all of the logarithms of the integers are irrationals, i.e., numbers which are incommensurable with the unit. Hence the incommensurables play a very important rôle in our mathematics.

We may say that Pythagoras discovered, invented or introduced the irrational into the number system. The number system as we know it contains also a *zero*, the *negative* numbers and the *imaginary*

or *complex* numbers. Strangely enough, it was not until 14 centuries later that the zero was introduced while the negatives and imaginaries came in 8 to 9 centuries later yet. In the language of the great German mathematician Kronecker (1823–1891), one of the great mathematicians of all time, "God made the positive integers; all the rest of the numbers were made by man." No one knows when or by whom the fractions were introduced. But to Pythagoras must be given credit for the introduction of the incommensurables or irrationals.

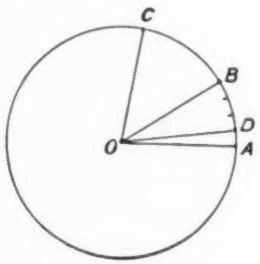
Thus the incommensurable forced its way into mathematics. The theory of proportion, built up so thoroughly and so carefully, must now be amended to allow for this case. Let us now consider the methods that are employed in attacking this case.

The two methods that are commonly employed are

1. Theory of limits;
2. The method of reductio ad absurdum.

I shall illustrate both methods on the same problem.

THEOREM. *In the same circle or in equal circles two central angles are in the same ratio as their intercepted arcs.*



CASE I. *When the arcs are commensurable.*

Let arc AD be a common measure which is contained m times in the arc AB and n times in arc BC. Then

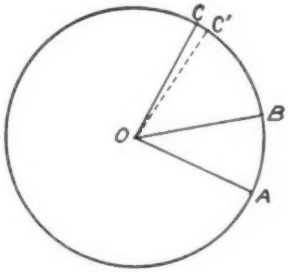
$$\frac{\text{arc } AB}{\text{arc } BC} = \frac{m}{n}.$$

Draw radii to the several points of division. Then angle AOB will be divided into m angles and angle BOC will be divided into n angles, all of which are equal.

$$\therefore \quad \frac{\angle AOB}{\angle BOC} = \frac{m}{n}.$$

$$\therefore \quad \frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

CASE II. *When the arcs are incommensurable.*



1. Proof by the theory of limits.

Let arc AB be divided into a number of equal parts and let one of these parts be applied to arc BC as the unit of measure. Since arcs AB and BC are incommensurable, a certain number of equal arcs will extend from B to C', leaving a remainder $C'C$ less than one of the equal arcs. Draw radius OC'. Since arcs AB and BC' are commensurable,

$$\frac{\angle AOB}{\angle BOC'} = \frac{\text{arc } AB}{\text{arc } BC'}.$$

Now let the number of subdivisions of the arc AB be indefinitely increased. The unit of measure will then be indefinitely diminished and the remainder CC', being always less than the unit, will approach the limit zero.

Then $\angle BOC'$ will approach the limit $\angle BOC$, and arc BC' will approach the limit arc BC. Hence, $\frac{\angle AOB}{\angle BOC'}$ will approach the limit $\frac{\angle AOB}{\angle BOC}$, and $\frac{\text{arc } AB}{\text{arc } BC'}$ will approach the limit $\frac{\text{arc } AB}{\text{arc } BC}$. Now

$\dfrac{\angle AOB}{\angle BOC'}$ and $\dfrac{\text{arc } AB}{\text{arc } BC'}$ are two variables which are always equal and they approach the limits $\dfrac{\angle AOB}{\angle BOC}$ and $\dfrac{\text{arc } AB}{\text{arc } BC}$, respectively. By the Theorem of Limits, these limits are equal.

$$(1) \qquad \therefore \qquad \frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

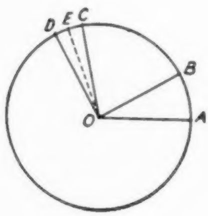
2. Proof by the method of reductio ad absurdum.

If the relation (1) is not true, there exists some arc BD, either greater than BC or less than BC, such that if substituted for BC in (1), the resulting relation is true. For the sake of definiteness let us suppose that BD is greater than BC. We then have

$$(2) \qquad \frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BD},$$

where $BD > BC$. Now suppose that the arc AB is divided into equal parts, each less than CD. If this arc be taken as the unit of measure and applied to arc BD, at least one point of division will fall between C and D. Let E be one such point and draw the radius EO. The arcs AB and BE being commensurable, we have by Part I of the Theorem:

$$(3) \qquad \frac{\angle AOB}{\angle BOE} = \frac{\text{arc } AB}{\text{arc } BE}.$$



From the relations (2) and (3) it follows that

$$\frac{\angle BOC}{\angle BOE} = \frac{\text{arc } BD}{\text{arc } BE}.$$

But this is impossible, since $\angle BOC$ is less than $\angle BOE$ while arc BD is greater than arc BE. Hence, our assumption that BD is greater than BC leads to a contradiction.

By entirely similar reasoning, it follows that the fourth term of the proportion (2) cannot be less than BC. Hence $BD = BC$.

Between these two methods of treating the incommensurable case, there is really very little choice. The first will appeal to one teacher, the second to another. As a student in High School, I had learned the first method of attack. Then when I became a student at college, I found that the author invariably used the second method. I recall very distinctly the reply that the instructor made when I remarked on that point. "The reason that this author does not use the Theory of Limits here is because to so many students the Theory of Limits is like a slot machine; the student presses a button, something turns somersaults inside and the answer shoots out; and the student is no wiser than he was before."

It seems that the more recent texts on plane geometry are omitting altogether the incommensurable case. This is in line with the movement which is sweeping all over the country to make mathematics texts easier. Perhaps this is right; I am not setting myself up as a judge of that. But it does seem a pity to delete from our texts a treatment of that fundamental discovery of Pythagoras which students of geometry have studied and understood for over 2000 years. Instead of being modern, we return to the work of 24 centuries ago, with this essential difference. The Greeks before Pythagoras did not treat of the incommensurable because they were not aware of its existence, while we modern Americans omit it because it is too difficult for *all* of our students to understand. In our ambition to give all our youth a High School and College education, when we find that our curriculum is too hard for them, we lower the standard of the curriculum. As a result our College and University standard is lower than that in England and on the continent. Doubtless there will be a reaction sooner or later and again there will appear in our texts on plane geometry that stiffening of the backbone, a treatment of the incommensurable.

Development of Mathematics in Secondary Schools of the United States

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Part II

MATHEMATICS DURING THE ACADEMY PERIOD*

The latin grammar schools had inherited certain aristocratic characteristics that existed in England at the time the American colonies were settled; these characteristics exhibited themselves in the practice of selecting pupils according to the rank and social status of their parents. From the conditions incident to frontier life they had acquired certain other characteristics such as limited means, limited facilities, limited purposes, and limited opportunities, all of which were manifested in the narrow curriculum of the period. As the population of these early colonies increased through immigration and birth, old communities broke up and migration westward began. The new settlements established in the wilderness were founded by people who had not known the religious zeal and oppression of the old country. With this shifting of population new interests in shipping and commerce began to replace the old interests in religion and agriculture. Such social and commerical expansion gradually led to a demand for a more liberal and democratic form of education, which demand was met by the organization of the academy. This new institution, to a certain extent an offspring of Philistinism, "came in to serve the broader need represented by those who would enter occupational pursuits without going to college, as well as those planning to continue their education. It was, in effect, an expression of expanding democracy."[1]

* The first part of this discussion appeared in the March issue of The Mathematics Teacher. The third part will appear in the May issue.

[1] L. V. Koos, *The American Secondary School,* Ginn and Co., (1927), p. 26. By permission of Ginn and Company, Publishers.

As early as 1743 Benjamin Franklin had formulated plans for the establishing of the first academy. Although the time was ripe for a new plan of education it was not until 1749 that Franklin was able to gain any recognition of his scheme for a school in which the pupils should learn those things that were likely to prove most useful and cultural and in which due consideration should be given to the different positions for which they wished to prepare.[2] In 1751 Franklin's Academy opened in Philadelphia, with three schools, English, Latin, and Mathematics, each under its own master. In 1754 a fourth school, the Philosophical, was added, and this resulted in the reincorporation of the institution in which the Latin and Philosophical Schools were spoken of as the College and the other two as the Academy. The real academy movement, however, had its beginning in 1800, went through its most rapid development between 1826 and 1835, and suffered a rapid decline after the year 1860.

Regardless of the momentous changes in the religious, political, and social conditions of the country in the latter part of the eighteenth century and the consequent new educational demands, the colleges remained bound in the fetters of the old traditions. The academy in meeting these new demands of the times gradually encroached upon the college and its educational program. The natural consequence of this was the material increase in college entrance requirements. At the close of the eighteenth century the only mathematics required for admission to college was a knowledge of the rules and processes of vulgar arithmetic. At Harvard the requirements for admission, as prescribed by the statutes of 1807, included the rules of arithmetic dealing with simple and compound notation, subtraction, multiplication, and division, together with reduction and the single rule of three. In 1820 the following addition was made to the list of subjects required for admission: "Algebra to the end of simple equations, comprehending also the doctrines of roots and powers, arithmetical and geometrical progression." Columbia followed with algebra in 1821, Yale in 1847, and Princeton in 1848. Candidates for admission to Harvard College in 1844 were examined in the following mathematical subjects: "Davies' and Lacroix's arithmetic, Euler's *Algebra* or Davies'

[2] Paul Monroe, *Principles of Secondary Education*, The Macmillan Co., New York, (1915), p. 54.

First Lessons in Algebra to the extraction of the square root, and an *Introduction to Geometry and the Science of Form*, prepared from the most approved Prussian Text Books; to VII of Proportions." Geometry was made an entrance requirement at Yale in 1856, at Princeton, Michigan, and Cornell in 1868, and at Columbia in 1870. The admission requirements for Harvard in 1870 included higher arithmetic, elements of algebra through quadratic equations, elementary plane geometry, and logarithms; at the same time Yale specified "higher arithmetic including the metric system of weights and measures; Day's Algebra to quadratic equations; and the first two books of Playfair's Euclid." The first, third, and fourth books of Davies' *Legendre* or Loomis's *Elements of Geometry* was acceptable as a substitute for Playfair's *Euclid*. Princeton's requirement list included arithmetic, algebra (to quadratic equations), and geometry (first book of Euclid or its equivalent). The requirement for admission to the freshman class of Columbia in 1869 embraced four books of Legendre's *Geometry* and a knowledge of the metric system of weights and measures in addition to arithmetic and algebra. Students entering Cornell had to offer arithmetic, algebra to quadratic equations, and plane geometry; while the University of Michigan examined candidates for admission to the classical course in arithmetic to include the fundamental rules, common and decimal fractions, percentage, proportion, involution and evolution, algebra to quadratic equations, and geometry to consist of the first four books of Davies' Legendre or the equivalent.[3]

The curriculum of the academy was much broader in its scope than was that of the Latin grammar school and, in general, less dominated by college entrance requirements. An examination of the annual reports made to the Board of Regents of the state of New York reveals that one hundred and forty-nine subjects appeared on the academy curricula during the period from 1787 to 1870, and among them the following mathematical subjects: arithmetic, algebra, astronomy, bookkeeping, conic sections, civil engineering, plane geometry, analytic geometry, leveling, logarithms, mapping, mensuration, navigation, nautical astronomy, statistics, surveying, and trigonometry.[4] In the period from 1826 to 1840,

[3] E. C. Broome, *A Historical and Critical Discussion of College Entrance Requirements*. The Macmillan Co., New York, (1903), pp. 41–53.

[4] P. Monroe, *op. cit.*, p. 58.

during which the largest number of new subjects appeared, algebra, geometry, and surveying were found among those subjects which attained a prevalency of seventy-five to one hundred per cent.[5]

Indications of the rapid growth of the academy and of the increasing importance of mathematics as a part of its curriculum may be observed in a study of the educational records of representative states. Massachusetts showed rapid development in the period from 1800 to 1860; these schools were both preparatory and finishing schools and offered a wide range of subjects including practical geometry, arithmetic, and algebra. In the Wilmington Academy, Wilmington, Delaware, the junior class studied all the rules of arithmetic, the four rules of algebra and the methods of solving simple equations, and the first book of Euclid. The senior class continued the study of Euclid in addition to trigonometry (plane and spherical), surveying and navigation, "the principles of astronomy and the Newtonian system, the solution of quadratic equations, and the principles of conic sections."[6] Another innovation of this new system of education was the extension to girls of the opportunity to attend school and we find that in December, 1831, the Ladies High School at Providence, Rhode Island, gave examination in arithmetic, algebra as far as quadratic equations, and plane geometry.[7]

Unlike the Latin grammar school the academy was not confined to the New England states. The South, soon after the Revolution, gave popular favor to this new educational institution. As in the East there was rather rapid expansion accompanied by a broadening of the curriculum. The academies of Virginia taught conic sections, higher mathematics, and the sciences.[8] In Tennessee the Manhattan School, of Maury County, included in its course of study arithmetic, geometry, trigonometry, surveying, navigation, conic sections, practical gunnery, practical astronomy, and the use of Hadley's octant and Davis's quadrant. Records for 1821 of the

[5] W. J. Gifford, *Historical Development of the New York State High School System*, J. B. Lyon Co., (1922), p. 20.

[6] L. P. Powell, *History of Education in Delaware*. Government Printing Office, Washington, D. C., (1893), p. 46.

[7] Henry Barnard, *The American Journal of Education*, H. C. Brownell, Hartford, Connecticut, (1858), Vol. 5, p. 21.

[8] C. J. Heatwole, *History of Education in Virginia*, The Macmillan Co., New York, (1916), p. 127.

Mathematics Seminary in Nashville show arithmetic, geometry, algebra, trigonometry, mensuration, and surveying. The Woodward School of Franklin in 1833 offered courses in algebra, geometry, logarithms, surveying, navigation, mensuration of heights and distances, surfaces and solids, and conic sections, while the Paris Female Institute included in its curriculum work in fractions, mensuration, extraction of roots, geometry, and projection of maps.[9] Records reveal programs very similar to these for the schools of Mississippi, Texas, and North Carolina.[10]

The curricula of these various institutions ranged from the simple elementary courses to rather advanced courses with mathematics forming an important part. The particular attention which was given to this subject was due in part to the possible practical applications, but principally to the idea of mental discipline which occupied a very prominent place in educational thought throughout these early years of American education and exerted profound influence upon curriculum making and methods of teaching. The attitude prevalent in the minds of text-book writers and teachers of that period is well stated in the words of Joseph Ray:

> The object of the study of mathematics is twofold—the acquisition of useful knowledge and the cultivation and discipline of the mental powers. A parent often inquires, 'Why should my son study mathematics? I do not expect him to be a surveyor, an engineer, or an astronomer.' Yet the parent is very desirous that his son should be able to reason correctly, and to exercise, in all his relations in life, the energies of a cultivated and disciplined mind. This is, indeed, of more value than the mere attainment of any branch of knowledge.[11]

The more extreme views on mental discipline did not go unchallenged, especially in the middle west. The pioneer life made it necessary for a man to rely upon his ability to develop the natural resources of this new country. This called for a type of education

[9] H. E. Ward, "Academy Education in Tennessee Prior to 1861," Unpublished Master of Arts Thesis, George Peabody College for Teachers, (1926), pp. 57–61.

[10] J. G. Warwick, "History of the Rise and Fall of Academies in Mississippi," Unpublished Master of Arts Thesis, George Peabody College for Teachers, (1927).

F. Erby, *Development of Education in Texas*, The Macmillan Co., New York, (1925).

C. L. Coon, *North Carolina Schools and Academies*, Edwards and Broughton, Raleigh, N. C., (1915).

[11] Joseph Ray, *New Elementary Algebra*, American Book Co., New York, (1848), p. III.

which placed more emphasis on the utilitarian values than on the cultural. President Pickett, of the College of Teachers, indicated, in an address in 1848, some of the main issues:

> One proposes to abandon the old mode, as it is called; to make way for a new; to knock down all the landmarks of time and wisdom, to erect them afresh; another, that Colleges and the higher order of schools are unnecessary, nay dangerous, because in them, an aristocratic spirit is cherished; a third, that the physical and mathematical sciences receive too much attention, as one may live a life without having a use for their application; a fourth, that the Latin and Greek languages are useless, and that the modern languages ought to take their place; a fifth, that every study should be of a utilitarian character; a sixth, that no kind of education is useful, except the specific kind that teaches a man to make dollars and cents, and hoard them for his own gratification; and finally, there comes one who declares himself the devoted servant of the people, and argues, that youth spend too much time in schools and college halls, and that by the new plan everything in knowledge and morals may be obtained without study or labor by scholars, and the whole of them too, in a few months.[12]

Mathematics appears to have weathered the storms without the loss of any considerable amount of prestige. Although the number of mathematical subjects in the curriculum remained fairly uniform throughout the life of the academy, the nature and purpose of instruction underwent a considerable change. The English influence predominated in the American schools from the time of the Revolution until 1820. Many of the instructors in the American colleges and academies had received their training in England and the majority of the texts were either English editions or copied rather closely from English authors. Hutton's *Mathematics*, Bonnycastle's *Algebra*, and Playfair's *Euclid* are three of the most frequently mentioned texts of this early period.[13]

The period of French influence may be dated from the appointment of Claude Crozet, who had been trained in the Polytechnic School of Paris, as professor of Mathematics at the United States Military Academy in 1817. For a period of approximately fifty years from this date texts by French authors were the most popular for general use, although they never entirely displaced the use of English texts. John Farrar, who was selected for the chair of mathematics and natural philosophy at Harvard, in 1818 pub-

[12] A. O. Hansen, *Early Educational Leadership in the Ohio Valley*, Public School Publishing Co., Bloomington, Ill., (1923), p. 70.

[13] F. Cajori, *The Teaching and History of Mathematics in the United States*, Government Printing Office, Washington, D. C., (1890), pp. 55–56.

lished an *Introduction to the Elements of Algebra*, selected from the *Algebra* of Euler, and a translation of Lacroix's *Algebra;* he also published a translation of Lacroix's *Arithmetic.*[14] Charles Davies, professor of mathematics at West Point (1823–1837), published Brewster's *Translation of Legendre* (1828), and Bourdon's *Algebra* (1834).[15]

Contemporaneous with the influx of French mathematics there was a revival of interest in elementary education under the influence of the Pestalozzian school. "The first fruit of Pestalozzian ideas in the teaching of arithmetic among us was Warren Colburn's *Intellectual Arithmetic Upon the Inductive Method of Instruction.*"[16] This text published first in 1821, marked the beginning of a new epoch in arithmetic and the teaching of arithmetic as the student was now supposed to replace the studying and memorizing of rules that he did not always fully understand by the setting up of his own rules as generalizations of his actual experiences.[17] The pupil was introduced to the new topics by means of practical problems and questions, the order of presentation always being from the concrete to the abstract. Colburn published a more advanced book in 1822 as a "Sequel" to his "First Lessons." This text was intended for use in the secondary schools and consisted of topics usually found in the earlier advanced arithmetics except for the omission of the rule of three, position, and powers and roots. The method of presentation was similar to that of the elementary text.[18]

The period from 1821 to 1857 was one of rapid development for the study of arithmetic. One hundred and ninety-five texts on this subject were published during these years; of this number forty-six were designed for the use in colleges and academies, and four were direct translations from the French.[19] From 1860 to 1892 there was no essential change in aim or content and few modifica-

[14] F. Cajori, *ibid.*, p. 128.

[15] F. Cajori, *ibid.*, p. 120.

[16] F. Cajori, *ibid.*, p. 106.

[17] Walter S. Monroe, "Warren Colburn on the Teaching of Arithmetic," *Elementary School Teacher*, Vol. 12, (1911–12), p. 466.

[18] Walter S. Monroe, *ibid.*, Vol. 13, (1912–13), pp. 239–246.

[19] J. M. Greenwood, "American Textbooks on Arithmetic," Annual Report of the Commissioner of Education, Bureau of Education, Washington, D. C., (1897–98), Vol. I, pp. 796–868.

tions in the methods of teaching.[20] The arithmetics of this latter period were combinations of the old, as found in colonial arithmetics, and the new, as found in the works of Colburn. The old arithmetics generally rejected reasoning, Colburn's arithmetics rejected rules and encouraged reasoning; the texts of the period 1860–1892 gave rules but at the same time gave demonstrations and encouraged the pupils to think.

The combined influences of the English and French mathematicians was just as noticeable in the teaching of algebra as in geometry and arithmetic. Pike's arithmetic contained an introduction to the study of algebra. John Gough attached an appendix to his practical arithmetic in which he discussed a few of the fundamentals of algebra. Other English editions were those of Webber (1801) Hutton (1801), Bonnycastle (1806), Simpson (1809), and Farrar (1818). Warren Colburn, who published his *Introduction to Algebra upon the Inductive Method of Instruction* in 1832, advanced new ideas as to methods of instruction in algebra. His idea was to make the transition from arithmetic to algebra as gradual as possible. As the learner was expected to derive most knowledge from solving the problems himself, the explanations were made as brief as were consistent with giving what was required. The problems were designed to exercise the learner in reasoning instead of making him a mere listener.[21] Two of the most popular algebra texts of this period were those of Charles Davies and Joseph Ray. While Davies stated in his introduction that Bourdon's *Algebra* had been closely followed, Ray claimed reference to many different sources in the construction of his text.

The academies broke away from the rigid and formal procedure of the Latin schools and showed little aversion to innovations. This is evidenced by the introduction of the Lancastrian monitorial system of instruction; by the adoption, in many cases, of the Fellenberg system of manual labor schools; by the embodiment of the Pestalozzian methods in the new texts; and, by the closer correlation of studies with actual needs and experiences in the teaching of natural sciences through demonstrations.[22] In spite of the many

[20] Walter S. Monroe, "Arithmetic as a School Subject," Bureau of Education Bulletin #10, (1917), Washington, D. C., p. 90.

[21] H. G. Meserve, "Mathematics One Hundred Years Ago," The *Mathematics Teacher*, Vol. 21, (1928), p. 339.

[22] Paul Monroe, *op. cit.*, p. 60.

changes there still remained too much memorizing and mere recitation with little or no opportunity for questions by the students.[23]

Geometry, before the nineteenth century, was taught dogmatically and the students merely memorized and worked by rule. It is hard to say just when formal demonstration of proofs took its place among the methods of geometrical instruction, but record of a public examination of 1834 relates the incident where the students "were standing before the blackboard *demonstrating* intricate problems; showing conclusively that *A*, *B*, *C*, equalled *D*, *E*, *F*; and then, by cabalistic figures, proving that 'plus' and 'minus' if properly managed would come out right at the last."[24] Although there remain traces of the dogmatic method of instruction in geometry, since the middle of the nineteenth century there has been a gradual change toward more emphasis on original exercises and construction with more attention given to intuitive geometry. These innovations in the teaching of geometry were accompanied by similar improvements in algebra and arithmetic. Many indications are found of attempts to improve methods of instruction and arrangement of subject matter, but for carefully planned scientific approach to such problems we are forced to turn to the high school period which followed.

[23] Timothy Dwight, "How I Was Educated," *Forum*, Vol. 2, (1886), p. 258.
[24] Henry Barnard, *American Journal of Education*, Vol. 5, (1858), p. 25.

"The progress of science, therefore, has been closely bound up with an increasing definiteness of terminology. The ideal terminology for this purpose has been found to be *mathematical symbols*. For, while even highly standardized words may have different shades of meaning to different minds, the symbol 205 means *exactly the same to everyone*. Likewise, the mathematical technique of *expressing relationships between ideas* is the most *highly perfected and standardized* that we have developed. So largely has science been dependent upon this development that we find the advancement of science largely contingent upon the development of mathematics. It is on the symbolic uniformity of mathematics that we rely for further power to describe the uniformities in the behavior of phenomena. The description of these uniformities constitutes the essence of science. In the description of *social phenomena* very little use has as yet been made of the language of science, namely mathematics."—"Social Research." Geo. A. Lindberg. pp. 55, 56.

Mathematics: A Forgotten Subject in Present-Day Schools

By Ira Russell Glover
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It is apparent to anyone who looks into the matter that mathematics and the teaching of it in the United States are in a slump, and that conditions hold no promise of improvement. If the present unsatisfactory state of the subject were a new thing, we could hope that interest in it might revive with a general revival of economic prosperity and educational expansion. Such, however, is not the case; the decline has been gradual and extended, in fact, over more than a decade. Mathematics is being neglected—relegated to a position of secondary importance which it certainly does not merit—in secondary school curricula, and, to a slightly lesser degree, in that of the colleges. For all of this, a variety of reasons doubtlessly can be found. Nevertheless, we, the mathematics teachers of the high schools and colleges, are largely to blame. We have not advertised; we have modernized neither our subject matter nor our teaching methods; we have stood passively by and allowed other courses to crowd ours out of the schools. Above all, we have not done our share in the organizing and administering of schools and colleges; we have especially neglected the teachers' colleges. We have permitted others to prescribe the requirements for teachers of mathematics and to set up the curricula for their training. The net result is that many people who try to teach our subject are not qualified to do it. And we should not be surprised at what has happened; it was inevitable. Of course, satisfactory outcomes can not be expected from any such procedures. So mathematics has been cheapened in the eyes of the general public and the study of it lessened; and all scientific work has suffered as a natural consequence. Certainly this could have been foreseen and forestalled, but even now, little is being done toward remedying it.

State requirements[1] for teachers of mathematics are often lower than for other regular academic subjects.[2] The quantity of col-

[1] Compiled from a study of the standards and requirements for high school teachers of all the states of the United States.

[2] West Virginia requires sixteen semester hours of mathematics; Maryland, eighteen. But both of these states require more hours for other subjects. See bulletins of the Departments of Education of these states.

lege credit in mathematics required varies from none to eighteen semester hours.[3] Only a very few states prescribe definite courses. Anything from shop mathematics to astronomy will usually be accepted for certification. This is true even in states which rank rather high educationally.[4] A minor from an accredited college or teachers' college will suffice in states which require no definite number of hours. In the majority of the better colleges, a minor varies from ten to twenty semester hours. Only occasionally is it more than twenty.

"What good is it?" is a question often heard in any discussion of mathematics; and school people often ask it. Knowing little or nothing of the subject, they fail to appreciate its value. Naturally such an attitude must have its effect upon students. "Oh, what is the use of taking mathematics? It will not help you." How often have you heard that or similar remarks from capable students? But we let things of this sort continue, while many of us console ourselves with the thought that good students alone should take mathematics.[5] While this is probably true, the fact remains that very few students, good or otherwise, are taking mathematics. Its study is being generally neglected.

If we consider the curricula set up by our teachers' colleges, we can see that the work offered is nothing but a dilution and an abridgement of the traditional courses of the liberal arts institutions. About the only change in the subject matter that can be noticed is that less of it is required. There has been very little attempt at adapting it to the needs of secondary school teachers.

The contents of the courses are slightly variable. In some schools no credit is given for arithmetic of the usual high school variety—commercial and industrial; but others give credit for arithmetic which appears to be little, if any, above the elementary level. The teachers' course in arithmetic seems to be largely materials and methods. Freshman mathematics is generally designed for students who expect to major in non-mathematical fields; but anyone is

[3] Pennsylvania requires eighteen semester hours; Connecticut at present only requires six, but they plan to more than double this requirement soon. A few of the Southern states still license by examination.

[4] See Phillips, *A Graphic View of our Schools*. Houghton Mifflin Co., New York, 1927. Also see bulletins of State Departments of Education.

[5] Many educators are of the opinion that no one with an I.Q. under 100 should try to study mathematics. Thorndike, *Psychology of Algebra*.

A Typical Teachers' College Curriculum for the Training of High School Mathematics Teachers[6]

Subject	*Semester hours[7]*
Arithmetic	2–6
Freshman Mathematics	2–4
College Algebra	2–6
Analytic Geometry	4–6
Trigonometry	2–5
Geometry (plane and solid)	2–5
College Geometry	2–3
Calculus	4–8
History of Mathematics	2–3
Projective Geometry	2–3
Teaching of Mathematics	2–3

allowed to take it for credit; and it is commonly taken by those who plan to become mathematics teachers. College alegbra is the usual liberal arts course; the same thing is true of analytic geometry, trigonometry and calculus. Once in a while the trigonometry is a course combining plane and spherical trigonometry. The calculus is always very elementary. A few schools give college credit

Typical Curriculum of a Private or Denominational College which Prepares Secondary Teachers[8]

Subject	*Semester hours[9]*
College Algebra	3–5
Freshman Mathematics	2–4
Trigonometry	2–5
Solid Geometry	2–3
Analytic Geometry	4–6
Calculus	6–8
Modern Geometry	2–3
Surveying	2–4
Differential Equations	2–3
Materials and Methods in Mathematics	2–4

[6] Compiled from a detailed study of catalogues of such schools in all parts of the United States. Naturally there is bound to be some variation; but the one given given below is typical. The larger schools, including the land grant colleges and universities, usually offer more opportunity for selection; but their courses are not especially suited to the preparation of teachers.

[7] Semester hours allowed for these courses vary through a rather wide range.

[8] Compiled from catalogues of such institutions located in all sections of the United States.

[9] Semester hours credit and courses vary slightly in these institutions. The variation in hours credit is not so great as is found in the teachers' colleges.

on plane geometry when it is not offered for entrance. The geometry however, is most frequently solid geometry. College geometry is the same course, slightly abbreviated, as the modern geometry of the liberal arts college. Projective geometry is not commonly offered; but when it is, it is comprised of the usual subject matter. Teaching of mathematics and history of mathematics, as offered in the teachers' colleges, will usually be covered as materials and methods in the liberal arts schools.

The above courses differ little from those of the teachers' colleges, except that they generally contain more subject-matter—that is, they are more difficult to master. Sometimes the work is split up into separate courses to a greater extent than is here indicated. Taking everything into consideration, there should be little difference in the teaching of one who prepared in a liberal arts school and one who prepared in a teachers' college, given equal natural ability. Practice teaching is offered in both types of institutions, but is under the control of the education department.

It is plain, from a consideration of the above facts, that mathematics teachers are in too many cases inadequately prepared. The usual high school mathematics courses are elementary algebra, advanced algebra, arithmetic, plane geometry, solid geometry and general mathematics. Less frequently, and only in large schools, plane trigonometry and plane analytic geometry are offered.[10] The first six courses named are standard; they are what may be called secondary mathematics. Of course in vocational high schools, courses in applied mathematics are offered; but such schools are not a large portion of the total number of secondary schools. So it is evident that, while mathematics teachers do not get enough of any kind of mathematics, they particularly lack knowledge of the fundamentals—algebra and geometry. They study too little of the subject matter which they will have to teach. For teachers in the secondary schools, we need a body of subject matter built around the high school subjects as a core. One might, as things now are, become a teacher of mathematics without taking a single semester hour of geometry beyond high school. Many mathematics teachers have actually taken only three semester hours of college algebra in all of their training.

[10] Compiled from a study of high school curricula and state courses of study from all sections of the United States.

A radically different curriculum for the preparation and training of high school, normal school, teachers' college, and even of regular college and university teachers is here suggested. It contains much more of the kind of mathematics needed for secondary school teaching than is usually offered. Furthermore, it does not neglect the higher subjects; nor does it interfere with future plans for graduate work. Beyond a doubt, our graduate students lack, more than any other one thing, a sound knowledge of the tools of their profession—algebra and geometry. It is true that this curriculum contains far more material than those in use at present. But there is no other way out—we shall simply have to require more. Our teachers must be better prepared, and higher standards alone will bring better preparation.

The reader should remember, however, that the writer is offering this only as a suggestion; further study is needed. But any successful reorganization will probably of necessity have to be made along these lines.

Suggested Curriculum for the Preparation of Teachers of Mathematics

Subject	*Semester hours credit*
Arithmetic	6
Algebra	8
Geometry	8
Electives	10

The above comprises a total of thirty-two semester hours of rather specifically-required mathematics. The only variable is in the electives. This, however, is not a greater number of hours than most good colleges require for a major in any other subject. It does prescribe fairly rigidly; and that is one of its aims. It is intended to make certain that each teacher of mathematics will have a thorough knowledge of his subject. By that, if it achieves its end, it will eliminate one of the greatest deficiencies in the teaching of mathematics.

The geometry should consist of a course or of courses covering the whole field of geometry, but all of it should be integrated around secondary plane and solid geometry. The core of all of the algebra would be the secondary school offerings, but more algebra than is now given should be brought in—all of the "college algebra." The arithmetic should be work which would promote an understanding of this subject in all its phases, particularly its philo-

sophical aspects. The electives should come from the calculus and the other more advanced branches of mathematics. Skill should be neglected in none of these; but one of the principal aims ought to be thorough mastery of all basic principles and an appreciation of the unity of all mathematical study. Details will have to be worked out slowly. For teaching in institutions of higher learning, in addition to the above, at least a year of productive graduate study should be demanded. In addition, regulations should be firmly enforced.

When we get teachers who can teach mathematics as a living subject—that is, make it live and grow in the classroom—interest in the study of it will revive automatically. A thorough knowledge of the subject itself, and of the methods of teaching it, should help to make such teachers.

Leonard Euler

Born at Basel, April 15, 1707
Died at St. Petersburg, September 18, 1783

THE FACT that Euler was born at Basel and died at St. Petersburg suggests a connection with the Bernoulli family.* This was the case. Euler's father had studied under Jacques Bernoulli, Euler studied under Jean. At the suggestion of Daniel and Nicholas Bernoulli, Catherine I of Russia invited Euler to come to St. Petersburg in 1727, and in 1733, Euler succeeded Daniel Bernoulli as professor of mathematics in the Academy of Science there.

At the insistence of his father who was a Calvinistic clergyman, Euler studied theology and oriental languages as well as medicine, astronomy, and physics. He took his masters degree at Basel in 1723. In 1727 his essay on the proper masting of ships was awarded second place in a contest conducted by the French Academy of Sciences. From 1727 to 1741 and from 1766 until his death, he was in St. Petersburg. From 1741 until 1766 he was in Prussia at the invitation of Frederick the Great who had invited him to come to his court at the suggestion of D'Alembert. It was during this interval that an invading Russian army pillaged Euler's home, but the empress Elizabeth in apology compensated him for his loss.

Euler's work was done under great handicaps although he possessed a splendid constitution and a remarkable memory. In 1735 he lost the sight of one eye, evidently from overwork in the solution of a problem which had taken other mathematicians several months, but which Euler did in three days. It should be noted that Gauss, using improved methods, later solved it in less than an hour. Euler's comment on his calamity was "I'll have fewer distractions." Frederick the Great had reference to this handicap when he wrote to Voltaire about Euler as "un gros cyclop de géomètre." In 1766 when Euler had returned to Russia, a cataract formed on his good eye and he continued with his work with the assistance of a servant who had no knowledge of mathematics. In 1771, he lost much of his property by fire, but fortunately his manuscripts were saved. About the same time, he had an operation for the

* See the *Mathematics Teacher*, October and December, 1933.

cataract, but he abused his recovered vision and again became blind.

Euler's papers provided the Academy of St. Petersburg with material for twenty years of its publications after his death. Even then many of his papers remained unprinted. His method of work was to begin with a specific problem, to solve it if possible, and to then study the problems which grew out of the first one. An illustration of this occurs in his Königsberg problem.* In Königsberg, two islands and the mainland are connected by seven bridges—one island has a bridge to each bank of the river Pregel and a bridge to the other island; the second island has two bridges to each bank. The problem is to walk across each bridge once and only once and return to the starting point. The problem cannot be solved but the route can be traced if one takes two walks using two separate routes. From this problem Euler passed to the study of unicursal problems of many sorts. Another of Euler's recreational problems, also impossible of solution, is allied to the study of magic squares—to select thirty-six officers, one of each of six ranks from six regiments and to arrange these men in a solid square such that each row and each rank contains one man of each rank and one man from each regiment.

Euler's contributions to elementary mathematics include the use of capital letters for the angles of a triangle and the corresponding lower case letters for the opposite sides. He is responsible for the use of the notation $f(x)$ for a function of x, for the use of the letter e for 2.718. . . . , for s to represent the semiperimeter of a triangle, for Σ for summation, and i for the square root of minus 1. Euler defined logarithms as exponents where earlier writers had considered them as terms of an arithmetic progression, corresponding to the terms of a geometric progression. He studied the problem of the logarithm of a negative number and formulated the theorem that log n has an infinite number of values which are all imaginary unless n is positive in which case one and only one value is real. Euclid has showed that any number of the form $2^{p-1}(2^p-1)$ is a perfect number if 2^p-1 is prime. Euler showed that this form includes all even perfect numbers and apparently an odd number cannot be perfect.

The *Source Book in Mathematics* includes the following excerpts

* See W. W. R. Ball, *Mathematical Recreations and Essays*, 1919 ed., p. 170.

from Euler's work: Proof that every integer is the sum of four squares, Use of the letter *e* to represent 2.718 . . . . , On differential equations of the second order.

In commenting on Euler's work, Ball says "He created a good deal of analysis, and revised almost all the branches of pure mathematics which were then known, filling up the details, adding proofs, and arranging the whole in a consistent form.*

VERA SANFORD

* W. W. R. Ball, *Short Account of the History of Mathematics*, p. 39.